# Which Neural Net Architectures Give Rise to Exploding and Vanishing Gradients?

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- Fix $d \geq 1$ and $\mathbf{n} = (n_j)_{j=0}^{d}$ .
- $\mathfrak{N}(d, \mathbf{n})$ – depth $d$ ReLU nets with hidden layer widths $n_j$.
- $f_{\mathcal{N}}$ – function computed by $\mathcal{N} \in \mathfrak{N}(d, \mathbf{n})$

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- **A.** $\mathbb{E}\left[Z^K\right] = \exp\left(\Theta_K\left(\sum_j \frac{1}{n_j}\right)\right)$

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$$\left|\frac{\partial f_{\mathcal{N}}}{\partial \operatorname{Act}_{\beta}^{(j)}}\right| \in \{0, \infty\} \quad \Longleftrightarrow \quad \operatorname{Var}[Z] = \operatorname{Var}[\|\nabla f_{\mathcal{N}}\|^2] \gg 1.$$

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1. $\mu^{(j)}, \nu^{(j)}$ are symmetric around 0
2. $\mathrm{Var}[\mu^{(j)}] = 2/n_{j-1}$
3. $\nu^{(j)}$ has not atoms

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$$2 \exp\left(\frac{1}{2}\sum_{j=1}^{d-1}\frac{1}{n_j}\right) \leq \mathbb{E}\left[Z^2\right] \leq \exp\left(C\sum_{j=1}^{d-1}\frac{1}{n_j}\right).$$

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3. *For* $K < \min\{n_j\}$, *there exists* $c_K, C_K > 0$ *so that*
$$\exp\left(c_K\sum_{j=1}^{d-1}\frac{1}{n_j}\right) \leq \mathbb{E}\left[Z^K\right] \leq \exp\left(C_K\sum_{j=1}^{d-1}\frac{1}{n_j}\right).$$

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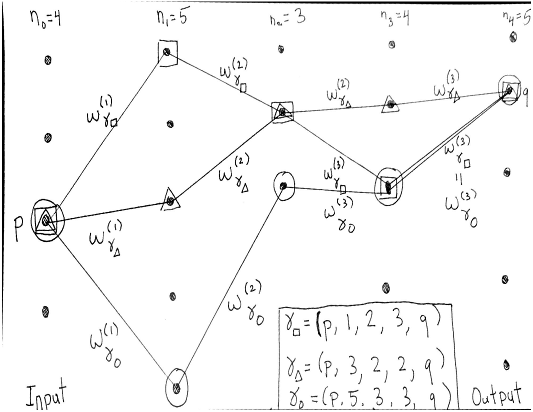

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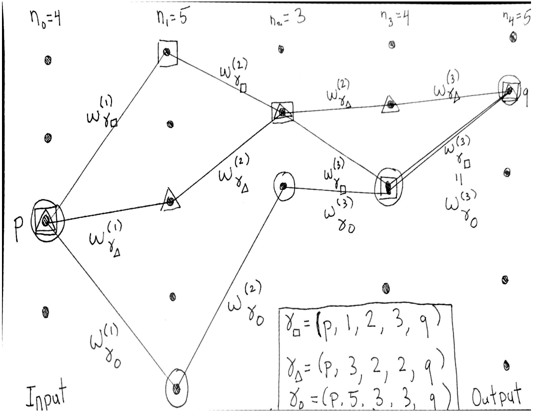


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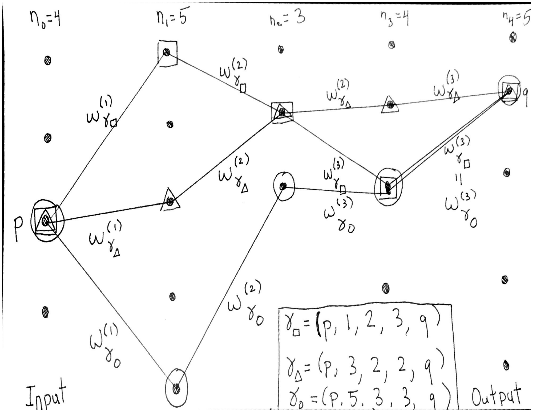


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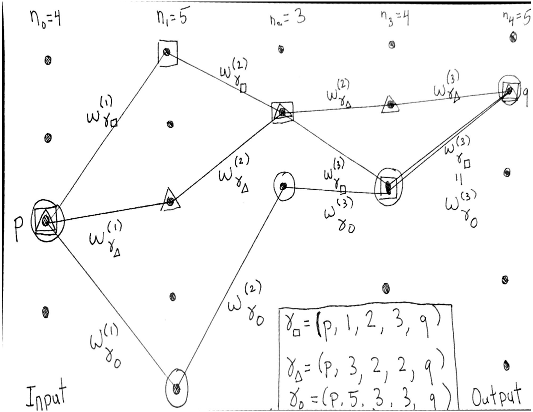


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- $\Gamma = (\gamma_\square, \gamma_\Delta, \gamma_O)$ has $\Gamma(2) = \{2, 3\}$ and $|\Gamma_{3,q}(5)| = 2$.

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*Let* $\mathcal{N} \in \mathfrak{N}_{\mu,\nu}(d, \mathbf{n})$. *Write* $Z_{p,q} = \partial (f_{\mathcal{N}})_q / \partial x_p$. *For every* $K \geq 0$,

$$\mathbb{E}\left[Z_{p,q}^{2K}\right] = \sum_{\substack{\Gamma = (\gamma_k)_{k=1}^{2K} \\ \gamma_k : p \to q}} \prod_{j=1}^{d} \left(\frac{1}{2}\right)^{|\Gamma(j)|} \prod_{\substack{\alpha \in \Gamma(j-1) \\ \beta \in \Gamma(j)}} \mu^{(j)}_{\left|\Gamma_{\alpha,\beta}(j)\right|},$$

*where*

$$\mu_r^{(j)} = \int x^r \, d\mu^{(j)}(x).$$

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## Remark

*The expression above is true for arbitrary connectivity and for convnets (when input is randomized).*

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- Use that $f_{\mathcal{N}}$ is a Markov Chain:

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- Use independence of neurons and symmetrize:

$$\mathbb{E}\left[\prod_{k=1}^{2K} w_{\gamma_k}^{(d)} \mathbf{1}_{\left\{\text{act}_{\gamma_k(d)}^{(d)} > 0\right\}} \;\middle|\; \text{Act}^{(d-1)}\right] = \prod_{\beta \in \Gamma(d)} \frac{1}{2} \mathbb{E}\left[\prod_{k=1}^{2K} w_{\gamma_k}^{(d)}\right].$$